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Abstract
We show that Green functions of second-order differential operators with singular or unbounded coefficients can have an anomalous behaviour in comparison to the well-known properties of the Green functions of operators with bounded coefficients. We discuss some consequences of such an anomalous short or long distance behaviour for a diffusion and wave propagation in an inhomogeneous medium.

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1. Introduction

A wave propagation with a constant speed is an approximation to the real situation when the wave propagates in a medium with its characteristics varying in space. A similar approximation is applied when considering a diffusion. In general, the speed of the diffusion can vary in space. If this variation is slow then one could believe that its effects are negligible. It has been observed some time ago (see [1] and references therein) that a diffusion in strongly inhomogeneous materials can be anomalous. This happens in particular with a heat convection in a turbulent medium [2, 3]. In general, it is rather difficult to investigate these problems because exact solutions are not available and approximations to equations with varying coefficients are not reliable. We discuss here the proper time method for a representation of the Green functions. We represent solutions of the equation

$$\mathcal{A}G_E = 2\delta \qquad (1)$$

for the Green function of an elliptic operator $\mathcal{A}$ in terms of its heat kernel. In this way the Green function is expressed by a diffusion. A solution of such an equation allows us to determine the wave propagation if we consider equation (1) either as an analogue of the Helmholtz equation for the propagation of monochromatic waves (then all the coordinates in (1) are spatial) or

continue analytically the solution G_E in one coordinate (identified with time) to imaginary values (we denote such a continuation by G_F). Then, the Green function G_F is known as the Feynman propagator. Its real part is a sum of the advanced and retarded propagators. In general, the Feynman propagator is not relevant to classical field theory. However, quantized electromagnetic field is necessary for a description of photons moving in an active medium which can be described as a dielectric or magnetic material [4–6].

If the coefficients of $\mathcal{A}$ are bounded regular functions then it is known [7] that there is a minor effect of varying coefficients on the local behaviour of the Green functions. We show that if the coefficients can grow or are singular then they have a profound effect on the Green functions. Then, the change of the behaviour of the Green functions has an impact on propagation of disturbances in the medium.

Let us mention some applications of our results to the wave equations in physics. Mechanical waves can be derived from Euler and continuity equations [8]. For example, the wave equation for the pressure p reads

$$\nu\partial_t^2 p = \nabla\rho^{-1}\nabla p \tag{2}$$

where ν is the compressibility and ρ is the density . An inhomogeneous equation whose solution is expressed by the Green function describes the effect of external forces acting upon the medium [9]. There is an equation similar to equation (2) for the scalar potential ϕ defined by the velocity $\mathbf{v} = -\nabla\phi$.

Next, we consider electromagnetic waves. There is a remarkable similarity between wave equations in electrodynamics and in fluid dynamics. It follows from the Maxwell equations that [10]

$$\epsilon\partial_t^2\mathbf{E} = \nabla\mu^{-1} \times \nabla \times \mathbf{E} - \partial_t\mathbf{J} \tag{3}$$

where μ is the magnetic susceptibility and ϵ is the dielectric permittivity. The equation for the magnetic field can be written in the form

$$\mu\partial_t^2\mathbf{H} = -\nabla\epsilon^{-1} \times \nabla \times \mathbf{H} + \nabla \times \epsilon^{-1}\mathbf{J}. \tag{4}$$

We could also introduce scalar ϕ and vector $\mathbf{A}$ potentials as

$$\mathbf{B} = \nabla \times \mathbf{A} \qquad \mathbf{E} = -\nabla\phi - \partial_t\mathbf{A}.$$

Then, in the static case it follows from Maxwell's equations that the scalar potential ϕ satisfies the equation

$$\partial_j\epsilon\partial_j\phi = -\rho \tag{5}$$

where ρ is the charge density. The vector potential $\mathbf{A}$ is a solution of the wave equation similar to equation (4) [5, 6].

The operator on the rhs of equations (3) and (4) is mixing the components of the electric and magnetic fields. It takes a simpler form if ϵ, μ and $\mathbf{J}$ do not depend on one coordinate x_D and we are looking for solutions (E_D or H_D) which do not depend on x_D. Then, equation (3) reads

$$\epsilon\partial_t^2 E_D - \partial_j\mu^{-1}\partial_j E_D = -\partial_t J_D. \tag{6}$$

Taking the Fourier transform in time of equation (6) we obtain the Helmholtz equation

$$-\epsilon\omega^2 E_D^\omega - \partial_j\mu^{-1}\partial_j E_D^\omega = -\mathrm{i}\omega J_D^\omega. \tag{7}$$

The quantum version of this model describes a quantum electromagnetic field interacting with a classical source. The correlation functions of the photon field and the scattering matrix for photons scattered on the source (or produced by the source) are described by

the Feynman propagator. Equations (6), (7) are reduced to a two-dimensional space. In the following section we discuss the scalar wave equation (2) in D space–time dimensions ($D - 1 = d$ space dimension) for greater generality. The scalar wave equation is often considered as a good approximation to the vector one. The Poisson-type equation (5) can be considered as a special case of equation (6) when the time derivative is neglected. In this paper we discuss quite unusual (unphysical) models for the space-dependent coefficients in equations (2)–(7). However, our results show that the conventional wisdom about spreading of forces in hydrodynamics and electrodynamics may require a modification in strongly inhomogeneous media.

2. Green functions of second-order elliptic differential operators

In application to the models (2)–(7) we discuss the following equation for the Green functions:

$$\left(B\partial_0^2 + \partial_j A \partial_j\right)G = 2\delta \tag{8}$$

where 0 can be either a space coordinate, an imaginary time or the real time. In the latter case B should be negative. Let us first discuss a special case $B = 1$. Then,

$$\left(\partial_0^2 + \partial_j A \partial_j\right)G_E \equiv 2\mathcal{A}G_E = 2\delta. \tag{9}$$

As an auxiliary tool for the wave equation we consider the diffusion equation

$$\partial_\tau P = \tfrac{1}{2}\partial_j A \partial_j P. \tag{10}$$

We consider the fundamental solution P_τ of equation (10) (the transition function of a Markov process). The operator $\mathcal{A}$ defined on the lhs of equation (9) is a non-negative symmetric operator in the Hilbert space of square integrable functions. Hence, it has the unique self-adjoint extension. We can define $\exp(-\tau\mathcal{A})$ and subsequently G_E as an integral kernel of

$$\mathcal{A}^{-1} = \int_0^\infty d\tau \, \exp(-\tau\mathcal{A})$$

(the equality holds true on the domain where $\mathcal{A}^{-1}$ makes sense; concerning Hilbert space methods for Green functions see [11]). $\mathcal{A}$ is a sum of two commuting operators. Hence, the kernel of $\exp(-\tau\mathcal{A})$ is a product of kernels of these operators. In this way the solution of equation (9) can be expressed by P_τ

$$G_E(x_0, \mathbf{x}; x_0', \mathbf{x}') = \int_{-\infty}^\infty d\omega \int_0^\infty d\tau \, P(\tau, \mathbf{x}, \mathbf{x}') \exp\left(-\frac{1}{2}\omega^2\tau\right)\exp(i\omega(x_0 - x_0'))$$

$$= \int_0^\infty d\tau \, P(\tau, \mathbf{x}, \mathbf{x}')(2\pi\tau)^{-\frac{1}{2}} \exp\left(-\frac{1}{2\tau}(x_0 - x_0')^2\right). \tag{11}$$

Let us denote by $\mathbf{q}_s(\mathbf{x})$ the diffusion process defined by the transition function P (equation (10)). Then, the Green function (8) can be expressed in the form

$$G_E(x_0, \mathbf{x}; x_0', \mathbf{x}') = \int d\omega \int d\tau$$

$$\times E\left[\exp\left(-\omega^2 \int_0^\tau B(\mathbf{q}_s(\mathbf{x})) \, ds\right) \exp(i\omega(x_0 - x_0'))\delta(\mathbf{q}_\tau(\mathbf{x}) - \mathbf{x}')\right] \tag{12}$$

where $E[\cdots]$ denotes the expectation value over the diffusion process $\mathbf{q}_\tau$ ([13, 14]) defined by the transition function P of equation (10). We cannot find a general solution of equations (8)–(10) but restrict ourselves to

$$A(\mathbf{x}) = k|\mathbf{x}|^\alpha \tag{13}$$

where k is a constant.

We can compute the transition function $P(\tau, \mathbf{0}, \mathbf{y})$ for a diffusion starting from a point $\mathbf{x} = \mathbf{0}$. Then, the solution for P in D space–time dimensions is

$$P(\tau, \mathbf{0}, \mathbf{y}) = K(k\tau)^{-\gamma} \exp\left(-\frac{b}{2k\tau}|\mathbf{y}|^{\beta}\right) \qquad (14)$$

where K is a normalization constant resulting from $\int \mathrm{d}\mathbf{y}\, P = 1$ and

$$\beta = 2 - \alpha \qquad b = 4(2 - \alpha)^{-2} \qquad (15)$$

$$\gamma = (D - 1)(2 - \alpha)^{-1}. \qquad (16)$$

The requirement of a normalization of P imposes the condition $\alpha < 2$.

We can calculate the variance of the diffusion path

$$E[\mathbf{q}_\tau(\mathbf{0})^2] = \int \mathrm{d}\mathbf{x}\, P(\tau, \mathbf{0}, \mathbf{x})\mathbf{x}^2 = C_1 \tau^{\frac{2}{2-\alpha}} \qquad (17)$$

where C_1 is a constant (which can be calculated and depends on the constants entering equation (14), it is finite if $\alpha < 2$). Hence, for $\alpha > 0$ we have a superdiffusive behaviour whereas for $\alpha < 0$ a subdiffusive one. There is no contradiction with the diffusion equation (10) which determines the diffusion coefficient as $\lim_{\tau \to 0} E[\tau^{-1} q_\tau^2]$ because $A(\mathbf{x}) \to 0$ as $\mathbf{x} \to \mathbf{0}$ in the superdiffusive case and $A(\mathbf{x}) \to \infty$ as $\mathbf{x} \to \mathbf{0}$ in the subdiffusive behaviour. In equation (11) we can calculate directly the τ integral if $\gamma + \frac{1}{2} > 1$. We obtain

$$G_E(x_0, \mathbf{0}; x'_0, \mathbf{x}) = C_2 \Omega_E(x_0, \mathbf{0}; x'_0, \mathbf{x})^{-\gamma + \frac{1}{2}} \qquad (18)$$

with a certain constant C_2 (depending on the parameters of equation (14)) where

$$\gamma - \frac{1}{2} = \left(D - 2 + \frac{\alpha}{2}\right)(2 - \alpha)^{-1} \qquad (19)$$

and

$$\Omega_E(x_0, \mathbf{0}; x'_0, \mathbf{x}) = (x_0 - x'_0)^2 + \frac{1}{k}\left(1 - \frac{\alpha}{2}\right)^{-2}|\mathbf{x}|^{\beta}. \qquad (20)$$

The function in the integral (11) is integrable at small τ for any γ if $\Omega_E > 0$ and for large τ if $\gamma + \frac{1}{2} > 1$, i.e. $D - 2 > -\frac{\alpha}{2}$. In the limiting case of $\gamma = 1/2$ the integral (11) is logarithmically divergent for large τ. The divergence means that the operator $\mathcal{A}$ in equation (9) is not positive definite. We can still define the Green function (9) in this limiting case (as a distribution) subtracting the divergent constant from the integral (11) (or imposing the condition $\int f \, \mathrm{d}\mathbf{x} = 0$ on the test functions f). Then, the Green function is proportional to $\ln \Omega_E$.

The special case of equation (9) is worth mentioning

$$\partial_j A \partial_j G_E = 2\delta. \qquad (21)$$

The solution is

$$G_E(\mathbf{x}, \mathbf{x}') = \int_0^\infty \mathrm{d}\tau\, P(\tau, \mathbf{x}, \mathbf{x}'). \qquad (22)$$

The integral (22) is convergent if $\gamma > 1$ ($D - 1 > 2 - \alpha$). Inserting the explicit solution (14) we obtain

$$G_E(\mathbf{0}, \mathbf{x}) = C|\mathbf{x}|^{-D+3-\alpha}. \qquad (23)$$

In the case $D - 3 = -\alpha$ again the logarithm will appear on the rhs of equation (23).

The limiting case $\alpha \to 2$ is also interesting because in this limit the formulae (14)–(20) lose their meaning. In order to study what happens for $\alpha = 2$ we have found the diffusion process whose sample paths give the transition function P (solving equation (10)). The

square of it has the form $(\mathbf{q}_\tau(\mathbf{x}))^2 = \mathbf{x}^2 \exp(c_1\tau + c_2 b_\tau)$ where b_τ is the Brownian motion. It follows that $\mathbf{q}_\tau(\mathbf{0}) = \mathbf{0}$ is the only solution starting from $\mathbf{0}$ (there is the zero solution also for $0 < \alpha < 2$ but in addition a non-trivial one (14)). This result can explain why the formulae (14)–(18) have no limit as $\alpha \to 2$. Now, instead of the power-law behaviour (17) we obtain $E[(\mathbf{q}_\tau(\mathbf{x}))^2] = \mathbf{x}^2 \exp(c\tau)$ (where $c > 0$ is a certain constant). We suggest that when $\alpha \to 2$ then the power-like relation between q and τ resulting from equations (14) and (17) is replaced by an exponential one following from an exponential increase of $\mathbf{q}_\tau^2$ (such a relation follows also from an explicit calculation of the transition function for the radial part of the process $\mathbf{q}_\tau$).

3. The wave propagation and other applications

We can continue equation (20) into real time. The result determines the Feynman propagator [12]

$$G_F(t, \mathbf{0}; t', \mathbf{x}) = C_2(\Omega_F + \mathrm{i}\epsilon)^{-\gamma + \frac{1}{2}} \tag{24}$$

where

$$\Omega_F = -(t - t')^2 + \frac{1}{k}\left(1 - \frac{\alpha}{2}\right)^{-2}|\mathbf{x}|^\beta. \tag{25}$$

Let us note some characteristic features of this behaviour. For $\mathbf{x} = 0$ we obtain

$$G_F(t, \mathbf{0}; t', \mathbf{0}) = C_3|t - t'|^{-2(D-2+\frac{\alpha}{2})(2-\alpha)^{-1}}. \tag{26}$$

$G_F(t, \mathbf{0}; t', \mathbf{0})$ can have arbitrarily high singularity (when $\alpha \to 2$). If $t = t'$ then

$$G_F(t, \mathbf{x}; t, \mathbf{0}) = C_4|\mathbf{x}|^{-D+2-\frac{\alpha}{2}}. \tag{27}$$

Hence, the Green function is integrable in $\mathbf{x}$. We can see that for $\alpha > 0$ the Green function is more singular than that with constant coefficients, whereas for $\alpha < 0$ it is less singular.

The Feynman propagator does not appear in the classical theory of wave propagation (except of Feynman–Wheeler electrodynamics) because it propagates the waves from the past and from the future as its real part is a sum of the advanced and retarded propagators

$$\operatorname{Re} G_F = \tfrac{1}{2}(G_A + G_R). \tag{28}$$

However, any solution G of equation (8) ($x_0 = \mathrm{i}t$) is a sum $G = G_F + U$ of G_F and a solution U of the wave equation

$$\left(B\partial_t^2 - \partial_j A \partial_j\right)U = 0.$$

Choosing a regular U with some boundary conditions (in particular at spatial infinity) we can obtain a Green function with the same singularity and appropriate boundary conditions.

The real part of G_F is a fractional derivative of $\delta(\Omega_F)$. If we send a signal from $\mathbf{0}$ at time t then its arrival at time t' at the point $\mathbf{x}$ can be calculated from the equation $\Omega_F(t, \mathbf{0}; t', \mathbf{x}) = 0$. Let us note that the wave front does not propagate with a constant speed but the velocity depends on time (or space). So, we obtain from Ω_F

$$\frac{\mathrm{d}|\mathbf{x}|}{\mathrm{d}t} = 2\beta^{-1}a^{\frac{1}{\beta}}\left(1 - \frac{\alpha}{2}\right)^{\frac{2}{\beta}}|t - t'|^{\frac{\alpha}{2-\alpha}}. \tag{29}$$

For the more general equation (8)

$$G_E(x_0, \mathbf{x}; x_0', \mathbf{x}') = \int \mathrm{d}\tau\, E\left[\exp\left(-(x_0 - x_0')^2\left(4\int_0^\tau B(\mathbf{q}_s(\mathbf{x}))\,\mathrm{d}s\right)^{-1}\right)\right.$$

$$\left. \times \left(4\pi\int_0^\tau B(\mathbf{q}_s(\mathbf{x}))\,\mathrm{d}s\right)^{-\frac{1}{2}} \delta(\mathbf{q}_\tau(\mathbf{x}) - \mathbf{x}')\right]. \tag{30}$$

The case $\alpha = 0$ in equation (13) ($A = k$) is also interesting. Then, $\mathbf{q}_s(\mathbf{x}) = \mathbf{x} + \mathbf{b}_s$, where $\mathbf{b}$ denotes the Brownian motion [13, 14] (in such a case we know $P(\tau, \mathbf{x}, \mathbf{x}')$ explicitly).

The expectation value $E[\cdots]$ in equation (30) cannot be calculated exactly. We have to resort to some approximations. If B is a bounded slowly varying function then approximately

$$B(\mathbf{q}_s(\mathbf{x})) \simeq B(\mathbf{x}) \simeq B(\mathbf{0}). \tag{31}$$

In such a case we obtain formula (20) for $G_E(x_0, \mathbf{0}; x_0', \mathbf{x}')$ with

$$\Omega_E(x_0, \mathbf{0}; x_0', \mathbf{x}') = B(\mathbf{0})^{-1}(x_0 - x_0')^2 + \frac{1}{k}\left(1 - \frac{\alpha}{2}\right)^{-2} |\mathbf{x}'|^\beta. \tag{32}$$

If B like A has a power-like behaviour

$$B(\mathbf{x}) = b|\mathbf{x}|^\sigma \tag{33}$$

then the approximation (31) cannot be justified. Let us note that from equation (14) it follows that

$$P(\tau, \mathbf{0}, \mathbf{y}) = P\left(1, \mathbf{0}, \tau^{-\frac{1}{\beta}}\mathbf{y}\right). \tag{34}$$

Hence,

$$\mathbf{q}_s(\mathbf{0}) \simeq \tau^{\frac{1}{\beta}} \mathbf{q}_{\frac{s}{\tau}}(\mathbf{0}). \tag{35}$$

Inserting this approximate equality (this equality is exact for the Brownian motion $\mathbf{b}$) into equation (30) and making the approximation

$$\int_0^1 B(\mathbf{q}_s(\mathbf{0}))\, ds \simeq E\left[\int_0^1 B(\mathbf{q}_s(\mathbf{0})\right] ds = C$$

we obtain

$$G_E(x_0, \mathbf{0}; x_0', \mathbf{x}) = KC^{-\frac{1}{2}} \int_0^\infty d\tau\, \tau^{-\gamma - \frac{1}{2} - \frac{\sigma}{4-2\alpha}} \exp\left(-\frac{1}{4C}\tau^{-1-\frac{\sigma}{2-\alpha}}(x_0 - x_0')^2 - \frac{b}{2k}\frac{|\mathbf{x}|^\beta}{\tau}\right). \tag{36}$$

We cannot compute the integral (36) exactly. Let us consider some special cases. If $x_0 = x_0'$ then

$$G_E(0, \mathbf{0}; 0, \mathbf{x}) \simeq |\mathbf{x}|^{-D+2-\frac{\alpha}{2}-\frac{\sigma}{2}}. \tag{37}$$

If $\mathbf{x} = \mathbf{x}'$

$$G_E(x_0, \mathbf{0}; x_0', \mathbf{0}) \simeq |x_0 - x_0'|^{-\nu} \tag{38}$$

where

$$\nu = (2(D-2) + \alpha + \sigma)(2 - \alpha + \sigma)^{-1}. \tag{39}$$

The case $\sigma = 0$ and $\alpha > 0$ could be considered as a realistic approximation to equation (2) describing the density of a fluid concentrated at $\mathbf{x} = 0$ (then $\rho(\mathbf{x})$ is decreasing from the origin as $|\mathbf{x}|^{-\alpha}$). In electrodynamics a scale-invariant dielectric permittivity is unrealistic. We could consider

$$\epsilon = \epsilon_0 + b|\mathbf{x}|^\alpha \tag{40}$$

where ϵ_0 is a constant. In application to equation (5) a source ρ produces the electric potential

$$\phi(\mathbf{y}) = -\frac{1}{2}(G_E\rho)(\mathbf{y}). \tag{41}$$

Hence, for a static point source located at $\mathbf{x} = 0$ it follows from equation (23) that

$$\phi(\mathbf{y}) = C_3|\mathbf{y}|^{3-D-\alpha} \tag{42}$$

is valid for small distances if $\alpha < 0$ and for large distances if $\alpha > 0$ ($D = 4$ in our world; equation (42) modifies the conventional $\frac{1}{r}$ Coulomb law). We obtain a similar formula from equation (4) for a magnetic field of a monochromatic wave.

The function B in equation (8) has a meaning of the magnetic susceptibility for equation (4). Hence, if

$$\mu = \mu_0 + b|\mathbf{x}|^\sigma \tag{43}$$

where μ_0 is a constant then $\int_0^\tau B(\mathbf{q}_s(\mathbf{0}))\, ds \simeq \mu_0\tau + c\tau^{1+\frac{\sigma}{\beta}}$. In such a case the approximations made at equation (30) hold true for small distances (resulting from small τ) if σ is negative and large distances if σ is positive. The same argument holds true when applied to equation (3) with the dielectric permittivity of the form (40). Then, the results of section 2 concerning scale-invariant coefficients apply to ϵ of the form (40) if we consider small distances in the case $\alpha < 0$ and large distances if $\alpha > 0$.

Finally, we consider QED in an inhomogeneous medium. The quantum electromagnetic field is defined by its time-ordered vacuum correlation functions which are determined by the Feynman propagator G_F. As an example of its applications we consider a scattering of photons on an external source J. The conventional formula for the S-matrix can be applied (see [15])

$$S = \exp\left(\frac{\mathrm{i}}{2}\int JG_F J\right)\exp\left(-\mathrm{i}\int A^{(-)}J\right)\exp\left(-\mathrm{i}\int A^{(+)}J\right) \tag{44}$$

where $A^{(-)}$ is the part of the vector potential linear in the creation operators, whereas $A^{(+)}$ is the annihilation part (positive energy solution of the wave equation). In particular, if the source is of the form $J(t, \mathbf{x}) = f(t)\delta(\mathbf{x})$ then the probability that the source will cause no photon emission (a preservation of the initial vacuum [15]) is

$$P(\mathrm{in}) = \exp\left(-\int \mathrm{d}\omega |\tilde{f}(\omega)|^2\tilde{G}_F(\omega)\right) \tag{45}$$

where tilde denotes the Fourier transform and $\tilde{G}_F(\omega)$ is the Fourier transform of the Green function (38). If $f(t) = \cos(\omega_0 t)$ then

$$\ln P(\mathrm{in}) \simeq \omega_0^{\nu-1}\delta(0) \simeq T\omega_0^{\nu-1} \tag{46}$$

where $\delta(0)$ is interpreted as the duration T of the signal J (this interpretation follows from the integral $\int JG_F J$ in equation (44) if T is large in comparison to the period $2\pi/\omega_0$). We can see that the inhomogeneity of the medium could be measured by a photon counting experiments showing whether $\omega^{\nu-1}$ is linear in ω or not.

4. Summary

The potential of a point charge is determined by the Poisson equation (5) with ρ as a δ source. The electric field can be obtained from equation (3). The spatial singularity and the decrease at infinity of the potential and of the electric field are determined by the behaviour of the Green functions. We have shown that this behaviour is not universal and may depend on the properties of the medium in which the charge is embedded. Analogous formulae are known for acoustics and fluid dynamics expressing the intensity of the waves produced by a given source. The laws governing the production of such waves will be modified if the fluid density or its compressibility is a growing or singular function of $\mathbf{x}$. In our examples there is always a distinguished point in $G_E(\mathbf{x}, \mathbf{x}')$ identified as an origin $\mathbf{x} = 0$ of the coordinate system. This point can be distinguished by the location of the source. The special role of the origin will disappear if coefficients in the differential operators are random variables invariant under translations (we discussed such random Green functions in [16]).

Acknowledgments

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